Second-order continuum traffic flow model

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A second-order traffic flow model is derived from microscopic equations and is compared to existing models. In order to build in different driver characteristics on the microscopic level, we exploit the idea of an additional phase-space variable, called the desired velocity originally introduced by Paveri-Fontana [Trans. Res. **9**, 225 (1975)]. By taking the moments of Paveri-Fontana's Boltzmann-like ansatz, a hierachy of evolution equations is found. This hierarchy is closed by neglecting cumulants of third and higher order in the cumulant expansion of the distribution function, thus leading to Euler-like traffic equations. As a consequence of the desired velocity, we find dynamical quantities, which are the mean desired velocity, the variance of the desired velocity, and the covariance of actual and desired velocity. Through these quantities an alternative explanation for the onset of traffic clusters can be given, i.e., a spatial variation of the variance of the desired velocity can cause the formation of a traffic jam. Furthermore, by taking into account the finite car length, Paveri-Fontana's equation is generalized to the high-density regime eventually producing corrections to the macroscopic equations. The relevance of the present dynamic quantities is demonstrated by numerical simulations. [S1063-651X(96)05911-9]

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I. INTRODUCTION

Theoretical descriptions of vehicular traffic are essentially based on two different viewpoints: one microscopic and the other macroscopic. In macroscopic theories, traffic is modeled as a continuum fluid. One of the earliest hydrodynamical models was proposed by Lighthill and Whitham [2]. It comprises a continuity equation and a speed-density relation, but does not take into account acceleration and inertia effects. Inspired by this model, a variety of higher-order models have been developed that try to incorporate these effects. A survey on continuum models is given in [3]. In order to make these models more accurate, one has to introduce additional terms in the evolution equation. Since most of these terms are based on heuristic considerations, their parameters have to be determined from experiment.

On the other hand, microscopic models treat each vehicle separately, and their motions are governed by laws derived from traffic observation. Here cellular automata are a widespread method to investigate traffic phenomena (see [4–7]). A different approach, similar to gas kinetics, has been chosen by Prigogine and Herman [8] and has been further developed in [1,9–13]. A very fruitful idea is the so-called desired velocity introduced by Paveri-Fontana [1]. The desired velocity reflects driver characteristics and allows one to distinguish the individual acceleration behavior of different drivers. Whereas in a recent paper Helbing [12] starts from a reduced version of Paveri-Fontana's equation, we treat the full Paveri-Fontana equation and derive a closed set of approximate moment equations. We start by recapitulating Paveri-Fontana's Boltzmann-like ansatz in Sec. II. In Sec. III the macroscopic model for "pointlike" vehicles is derived by taking moments of the microscopic equation. The hierarchy of moment equations is closed by neglecting third- and higher-order cumulants, thus leading to Euler-like traffic equations. Owing to the introduction of the desired velocity additional dynamical equations for the new macroscopic quantities, the mean desired velocity, the variance of the desired velocity, and the covariance of actual and desired speed are found. To close the system of equations no speed-density relation is needed here. The homogeneous solution is given and the characteristic velocities are calculated. In order to generalize our model to high densities we take into account the finite car length (Sec. IV). This results in corrections to the evolution equations and keeps the system from reaching infinitely high densities. The relevance of the additional equations is demonstrated by computer simulations in Sec. V. We find that a spatial variation of the variance of the desired velocity can cause the formation of a traffic jam.

II. BOLTZMANN-LIKE MICROSCOPIC MODEL

To overcome the problems of the original kinetic model given of Prigogine and Herman [8], Paveri-Fontana [1] suggested the following improved Boltzmann-like microscopic model. The essential idea is the introduction of an additional phase-space coordinate, the so-called desired velocity w. Let $g(x,v,w,t)$ denote the one-vehicle distribution function for vehicles with desired speed w in the phase space spanned by x, v, w, t, where $g(x,v,w,t)dx\,dv\,dw$ denotes the number of vehicles at time t, in position dx around x, and actual speed dv around v with desired speed dw around w. The road is assumed to be a one-dimensional unidirectional lane, but passing is allowed.

The *one-vehicle speed distribution function* $f(x,v,t)$ and the *one-vehicle desired speed distribution function* $f_0(x,w,t)$ are given by

$$f(x,v,t)=\int_0^{+\infty} dw\ g(x,v,w,t), \tag{2.1}$$

$$f_0(x,w,t)=\int_0^{+\infty} dv\ g(x,v,w,t). \tag{2.2}$$

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The vehicular concentration $c(x,t)$, the average velocity $\overline{v(x,t)}$, the average desired velocity $\overline{w(x,t)}$, and the flow $q(x,t)$ are then defined as

$$c(x,t)=\int_0^{+\infty}dw\int_0^{+\infty}dv\ g(x,v,w,t), \tag{2.3}$$

$$\overline{v(x,t)}=\frac{\int_0^{+\infty}dw\int_0^{+\infty}dv\ vg(x,v,w,t)}{c(x,t)}, \tag{2.4}$$

$$\overline{w(x,t)}=\frac{\int_0^{+\infty}dw\int_0^{+\infty}dv\ wg(x,v,w,t)}{c(x,t)}, \tag{2.5}$$

$$q(x,t)=c(x,t)\overline{v(x,t)}. \tag{2.6}$$

Higher-order velocity moments are defined by

$$m_{k,l}(x,t)=\int_0^{+\infty}dw\int_0^{+\infty}dv\ v^k w^l g(x,v,w,t),\quad k,l\in\mathbb{N}, \tag{2.7}$$

for example,

$$c=m_{0,0}, \tag{2.8}$$

$$c\overline{v^k}=m_{k,0}, \tag{2.9}$$

$$c\overline{w^l}=m_{0,l}. \tag{2.10}$$

With $\vec{x}=(x,v,w)$, the total local change of the phase-space density is given through a continuity equation [14,15]

$$\frac{\partial g}{\partial t}+\nabla_{\vec{x}}\cdot\left(g\ \frac{d\vec{x}}{dt}\right)=\left(\frac{\partial g}{\partial t}\right)_{\text{coll}}. \tag{2.11}$$

The term on the left-hand side describes the continuous streaming in phase space, while the term on the right-hand side is due to discontinuous motion in phase space, i.e., sudden changes of the velocities due to collisions.

In analogy to the scattering process in kinetic gas theory the following interaction process between vehicles has been introduced by Prigogine and Herman [8] and has been adopted by Paveri-Fontana [1]. When a fast car reaches a slow car it either passes or slows down to the velocity of the car in front. Now the following assumptions are made.

(i) The ''slowing'' down process has a probability $(1-P)$, where P denotes the probability of passing $0\leqslant P\leqslant 1$. If the fast car passes the slow one, its velocity is not affected.

(ii) The velocity of the slow car is unaffected by the interaction or by being passed.

(iii) Cars are regarded as pointlike objects, so the vehicle length can be neglected.

(iv) The slowing down process is instantaneous; there is no braking time.

(v) Only two-vehicle interactions are to be considered; multivehicle interactions are excluded.

(vi) One assumes ''vehicular chaos,'' i.e., vehicles are not correlated,

$$g_2(x,v,w,x',v',w',t)\simeq g(x,v,w,t)g(x',v',w',t), \tag{2.12}$$

where g_2 denotes the two-vehicle distribution function.

Some remarks are necessary. The probability of passing is usually chosen to be density dependent, for example, $P(c)=1-c/\hat{c}$ ($\hat{c}$ denotes the maximal density [8]), but additional velocity and variance dependences have been proposed in [9]. Vehicle length can be taken into account by choosing an approach similar to Enskog's theory for dense gases [16,17] and is considered in Sec. IV. The assumption of instantaneous interaction is approximately valid for processes where the slowing down time $\Delta\tau$ and the length $v\Delta\tau$ are short compared to the characteristic time and length scales involved. Having made the assumption of vehicular chaos, the theory is actually only valid for dilute traffic.

The term on the right-hand side of Eq. (2.11) is a collision integral analogous to the Boltzmann term

$$\left(\frac{\partial g}{\partial t}\right)_{\text{coll}}=f(x,v,t)\int_v^{+\infty}dv'(1-P)(v'-v)g(x,v',w,t)$$
$$-g(x,v,w,t)\int_0^v dv'(1-P)(v-v')f(x,v',t). \tag{2.13}$$

The first part describes the gain of the phase space element, i.e., vehicles with velocity $v'\geqslant v$ collide with vehicles with velocity v, while the second term describes the loss of the phase space element, i.e., vehicles with velocity v collide with vehicles with even slower velocity v'.

Considering the streaming term in Eq. (2.11), Paveri-Fontana models the acceleration by

$$\frac{dv}{dt}=\frac{w-v}{T}, \tag{2.14}$$

i.e., the drivers approach their desired speed exponentially in time, with time constant T. One can also choose T to be a function of c,v (see [9]). Additionally, it is assumed that no driver changes his desired speed, resulting in

$$\frac{dw}{dt}=0. \tag{2.15}$$

Setting $dw/dt\neq 0$ would allow one to change the individual desired speed, for example, owing to externally imposed speed limitations. Collecting all the terms, the Boltzmann-like kinetic equation now reads

$$\left(\frac{\partial}{\partial t}+v\ \frac{\partial}{\partial x}\right)g+\frac{\partial}{\partial v}\left(\frac{w-v}{T}\ g\right)$$
$$=f(x,v,t)\int_v^{+\infty}dv'(1-P)(v'-v)g(x,v',w,t)$$
$$-g(x,v,w,t)\int_0^v dv'(1-P)(v-v')f(x,v',t). \tag{2.16}$$

In the Prigogine-Herman equation the acceleration term is modeled by a relaxation term towards an equilibrium distribution. Unfortunately, this leads to inconsistences (see [1]).

Alberti and Belli [10] proposed two kinetic equations: one for the distribution of free cars and one for the distribution of queuing cars. Nelson [13], on the other hand, developed a kinetic equation where speeding up interactions are treated on the same line as traditionally has been done for the slowing down process. For this, he employed a generalized vehicular chaos hypothesis.

III. MACROSCOPIC MODEL

Integration of relation (2.16) over dv leads to

$$\frac{\partial}{\partial t} f_0(x,w,t)+\frac{\partial}{\partial x}[\overline{v(x,w,t)}f_0(x,w,t)]=0, \tag{3.1}$$

where $\overline{v(x,w,t)}$ is defined as

$$\overline{v(x,w,t)}=\frac{\int_0^{+\infty}dv\ vg(x,v,w,t)}{f_0(x,w,t)}. \tag{3.2}$$

Equation (3.1) is a continuity equation for each desired speed w separately, as a consequence of Eq. (2.15), i.e., no driver changes his desired speed.

Using the notation of Eq. (2.7), the moments in v are

$$\frac{\partial}{\partial t} m_{k,0}+\frac{\partial}{\partial x} m_{k+1,0}+\frac{k}{T}(m_{k,0}-m_{k-1,1}) =(1-P)(m_{k,0}m_{1,0}-m_{k+1,0}m_{0,0}), \tag{3.3}$$

where $m_{-1,1}$ is defined as

$$m_{-1,1}=0.$$

Equations (3.1) and (3.3) have already been derived in the original work of Paveri-Fontana. The equations for the moments in w are given by

$$\frac{\partial}{\partial t} m_{0,l}+\frac{\partial}{\partial x} m_{1,l}=0, \tag{3.4}$$

and as the general form for the mixed moments in v and w one finds

$$\frac{\partial}{\partial t} m_{k,l}+\frac{\partial}{\partial x} m_{k+1,l}+\frac{k}{T}(m_{k,l}-m_{k-1,l+1}) =(1-P)\int_0^{+\infty}dw\int_0^{+\infty}dv\ w^l g(x,v,w,t) \times\int_0^{v}dv'(v'^k v-v'^{k+1}-v^{k+1}+v^k v')f(x,v',t). \tag{3.5}$$

Unfortunately, the right-hand side of Eq. (3.5) cannot be given in a closed form in terms of higher-order moments without approximations. For $k=0$ and 1, Eq. (3.3) yields the usual continuity equation

$$\frac{\partial}{\partial t} c+\frac{\partial}{\partial x}(c\overline{v})=0 \tag{3.6}$$

and the momentum equation

$$\frac{\partial}{\partial t}(c\overline{v})+\frac{\partial}{\partial x}(c\overline{v^2})+\frac{c}{T}(\overline{v}-\overline{w})=(1-P)c^2(\overline{v}^2-\overline{v^2}), \tag{3.7}$$

respectively. For the mean desired velocity, Eq. (3.4) leads to

$$\frac{\partial}{\partial t}(c\overline{w})+\frac{\partial}{\partial x}(c\overline{vw})=0 \tag{3.8}$$

for $l=1$. The dynamic equations for the second-order moments are given by

$$\frac{\partial}{\partial t}(c\overline{v^2})+\frac{\partial}{\partial x}(c\overline{v^3})+\frac{2c}{T}(\overline{v^2}-\overline{vw})=(1-P)c^2(\overline{v^2}\,\overline{v}-\overline{v^3}), \tag{3.9}$$

$$\frac{\partial}{\partial t}(c\overline{w^2})+\frac{\partial}{\partial x}(c\overline{vw^2})=0, \tag{3.10}$$

and

$$\frac{\partial}{\partial t}(c\overline{vw})+\frac{\partial}{\partial x}(c\overline{v^2w})+\frac{c}{T}(\overline{vw}-\overline{w^2}) =(1-P)\int_0^{+\infty}dw\int_0^{+\infty}dv\ wg \times\int_0^{v}dv'(2v'v-v'^2-v^2)f'. \tag{3.11}$$

A. Closure of the macroscopic equations

Equations (3.6)–(3.11) represent a hierarchy of moment equations, each evolution equation involving moments of the next higher order. The equation for the mixed moments cannot be given in terms of a finite number of moments. To close the system of equations, one has to make an approximation in form of a closing assumption. One defines

$$\delta v:=v-\overline{v},\quad \delta w:=w-\overline{w}. \tag{3.12}$$

Then the higher moments are given in terms of cumulants as

$$\begin{aligned}
\overline{v^2}&=\overline{(\delta v)^2}+\overline{v}^2,\\
\overline{w^2}&=\overline{(\delta w)^2}+\overline{w}^2,\\
\overline{vw}&=\overline{\delta v\,\delta w}+\overline{v}\,\overline{w},\\
\overline{v^3}&=\overline{(\delta v)^3}+3\overline{(\delta v)^2}\,\overline{v}+\overline{v}^3,\\
\overline{v^2w}&=\overline{(\delta v)^2\delta w}+2\overline{(\delta v\,\delta w)}\,\overline{v}+\overline{(\delta v)^2}\,\overline{w}+\overline{v}^2\overline{w},\\
\overline{vw^2}&=\overline{(\delta w)^2\delta v}+2\overline{(\delta w\,\delta v)}\,\overline{w}+\overline{(\delta w)^2}\,\overline{v}+\overline{w}^2\overline{v}.
\end{aligned} \tag{3.13}$$

The second-order cumulants are abbreviated by

$$\Theta_{vv} := \overline{(\delta v)^2},$$

$$\Theta_{ww} := \overline{(\delta w)^2},$$

$$\Theta_{vw} := \overline{\delta v\, \delta w}. \tag{3.14}$$

Now, using the continuity equation (3.6) and definition (3.14), Eq. (3.7) transforms into the momentum equation

$$\partial_t \overline{v} + \overline{v}\,\partial_x \overline{v} = \frac{\overline{w} - \overline{v}}{T} - \frac{\Theta_{vv}}{c}\,\partial_x c - \partial_x \Theta_{vv} - (1-P) c \Theta_{vv}, \tag{3.15}$$

while Eq. (3.8) now reads

$$\partial_t \overline{w} + \overline{v}\,\partial_x \overline{w} = -\frac{\Theta_{vw}}{c}\,\partial_x c - \partial_x \Theta_{vw}. \tag{3.16}$$

If one now exploits Eqs. (3.15) and (3.16), one can rewrite Eqs. (3.9) and (3.10) in terms of cumulants as

$$\partial_t (c\Theta_{vv}) + \overline{v}\,\partial_x (c\Theta_{vv}) + 3c\Theta_{vv}\partial_x \overline{v} + \partial_x [c\overline{(\delta v)^3}] + \frac{2c}{T}(\Theta_{vv} - \Theta_{vw}) + (1-P)c\overline{(\delta v)^3} = 0 \tag{3.17}$$

and

$$\partial_t (c\Theta_{ww}) + \overline{v}\,\partial_x (c\Theta_{ww}) + 2c\Theta_{vw}\partial_x \overline{w} + c\Theta_{ww}\partial_x \overline{v} + \partial_x [c\overline{(\delta w)^2 \delta v}] = 0, \tag{3.18}$$

respectively.

Dealing with Eq. (3.11) is much more involved and therefore only sketched here. The left-hand side (LHS) leads to

$$[\text{LHS (3.11)}] = \partial_t (c\Theta_{vw}) + \overline{v}\,\partial_x (c\Theta_{vw}) + 2c\Theta_{vw}\partial_x \overline{v} + c\Theta_{vv}\partial_x \overline{w} + \partial_x [c\overline{(\delta v)^2 \delta w}] - (1-P)c^2 \overline{w}\Theta_{vv} + \frac{c}{T}(\Theta_{vw} - \Theta_{ww}). \tag{3.19}$$

The right-hand side (RHS) is abbreviated

$$[\text{RHS (3.11)}] = (1-P)(2\mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3). \tag{3.20}$$

The integrals $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ are calculated in Appendix A under the assumption of a local Gaussian distribution in v and w, i.e., the cumulants of third and higher order are neglected. The approximation of a Gaussian velocity distribution is justified by empirical data (see [18,19,9,20]). For the distribution in the desired velocity w we suppose that the number of drivers with desired velocity w is also normally distributed. From traffic considerations we expect a distribution with a long high-velocity tail, but as a first approximation a Gaussian seems to be sufficient. In Appendix A we have chosen a cumulant expansion to evaluate the integrals since this method can easily be extended to include third-order cumulants when proceeding to the derivation of Navier-Stokes-like equations. Thus one finds for the RHS

$$[\text{RHS (3.11)}] = -(1-P)\left(c^2 \overline{w}\Theta_{vv} + \frac{2}{\sqrt{\pi}} c^2 \sqrt{\Theta_{vv}}\Theta_{vw} \right). \tag{3.21}$$

Taking Eqs. (3.17)–(3.19) and (3.21), the assumption of negligible third- and higher-order cumulants yields

$$\partial_t \Theta_{vv} + \overline{v}\,\partial_x \Theta_{vv} + 2\Theta_{vv}\partial_x \overline{v} + \frac{2}{T}(\Theta_{vv} - \Theta_{vw}) = 0, \tag{3.22}$$

$$\partial_t \Theta_{ww} + \overline{v}\,\partial_x \Theta_{ww} + 2\Theta_{vw}\partial_x \overline{w} = 0, \tag{3.23}$$

$$\partial_t \Theta_{vw} + \overline{v}\,\partial_x \Theta_{vw} + \Theta_{vw}\partial_x \overline{v} + \Theta_{vv}\partial_x \overline{w} + \frac{1}{T}(\Theta_{vw} - \Theta_{ww}) = -(1-P)\frac{2}{\sqrt{\pi}} c\Theta_{vw}\sqrt{\Theta_{vv}}. \tag{3.24}$$

Equations (3.6), (3.15), (3.16), and (3.22)–(3.24) represent a closed system of evolution equations for the variables c, $\overline{v}$, $\overline{w}$, Θ_{vv}, Θ_{ww}, and Θ_{vw}.

We compare this system of equations with other continuum models. Similar to the models of [21,22,11,12], the mean velocity equation (3.15) has a relaxation term, but $\overline{w}$ becomes a dynamical quantity [see Eq. (3.16)] and similar to [12] no equilibrium speed-density relation is needed to close the system. In addition, coupled equations for the variance Θ_{vv}, the variance Θ_{ww}, and the covariance Θ_{vw} appear. In contrast to some of the above-mentioned models, no second-order space derivatives appear since up to now the model is just Euler-like.

B. Homogeneous solution

The homogeneous solution for the system of partial differential equations (3.6), (3.15), (3.16), and (3.22)–(3.24) is found to be

$$\overline{w} - \overline{v} = T[1-P(c)]c\Theta_{vv}, \tag{3.25}$$

$$\Theta_{vv} = \Theta_{vw}, \tag{3.26}$$

$$\Theta_{ww} = \Theta_{vw}\left(1 + T[1-P(c)]\frac{2}{\sqrt{\pi}} c\sqrt{\Theta_{vv}}\right). \tag{3.27}$$

Thus, given certain values for c, $\overline{v}$, and Θ_{vv}, the mean desired velocity $\overline{w}$, the variance Θ_{ww}, and the covariance Θ_{vw} are determined.

C. Characteristic velocities

The characteristic velocities yield locally the domain of influence [2]. Set

$$\vec{u} = (c, \overline{v}, \overline{w}, \Theta_{vv}, \Theta_{ww}, \Theta_{vw})^T. \tag{3.28}$$

Equations (3.6), (3.15), (3.16), and (3.22)–(3.24) can then be written as

$$\partial_t \vec{u} + \underline{C}\cdot\partial_x \vec{u} = \vec{f}, \tag{3.29}$$

with

$$\vec{f}=\left(0,\frac{\overline{w}-\overline{v}}{T}-(1-P)c\Theta_{vv},0,\frac{2}{T}(\Theta_{vw}-\Theta_{vv}),0,\times\frac{\Theta_{ww}-\Theta_{vw}}{T}-(1-P)\frac{2}{\sqrt{\pi}}c\Theta_{vw}\sqrt{\Theta_{vv}}\right)^{T} \tag{3.30}$$

and

$$\underline{C}=\begin{pmatrix} \overline{v} & c & 0 & 0 & 0 & 0 \\ \Theta_{vv}/c & \overline{v} & 0 & 1 & 0 & 0 \\ \Theta_{ww}/c & 0 & \overline{v} & 0 & 0 & 1 \\ 0 & 2\Theta_{vv} & 0 & \overline{v} & 0 & 0 \\ 0 & 0 & 2\Theta_{vw} & 0 & \overline{v} & 0 \\ 0 & \Theta_{vw} & \Theta_{vv} & 0 & 0 & \overline{v} \end{pmatrix}. \tag{3.31}$$

The characteristic velocities are now given by the eigenvalues of the matrix $\underline{C}$ [2]. One finds

$$\lambda_{1,2}=\overline{v}, \tag{3.32}$$

$$\lambda_{3,4}=\overline{v}\pm\sqrt{\Theta_{vv}}, \tag{3.33}$$

$$\lambda_{5,6}=\overline{v}\pm\sqrt{3\Theta_{vv}}. \tag{3.34}$$

Having found six real eigenvalues, Eq. (3.29) is a hyperbolic system.

IV. CORRECTIONS FOR EXTENDED VEHICLES

In this section we try to incorporate in analogy to [16,23,17] the fact that cars are not pointlike objects but have a spatial extension of l and require an additional safety distance τv, depending on the velocity of each individual driver and a reaction time τ. Two different effects are taken into account.

(i) Since the covolume of the cars, i.e., length plus safety distance, is now comparable with the total volume of the system, the volume where the center of any vehicle can lie is reduced and therefore the collision frequency is enhanced by a factor χ.

(ii) The common position x of the two colliding vehicles in the Boltzmann integral should be replaced by the actual positions of the centers of the two vehicles and the ‘‘cross section’’ should be taken at the actual position of the collision.

A. Factor χ

Denote by $d(\overline{v})=l+\tau\overline{v}$ the mean required length at a certain mean speed $\overline{v}$, with τ being the reaction time. The maximal density at this mean speed $\overline{v}$ is $\hat{c}(\overline{v})$,

$$\hat{c}(\overline{v})=\frac{1}{d(\overline{v})}=\frac{1}{l+\tau\overline{v}}. \tag{4.1}$$

The effective volume is then reduced by a factor $1-c/\hat{c}(\overline{v})$ and therefore the scattering probability is increased by a factor $\chi=[1-c/\hat{c}(\overline{v})]^{-1}$,

$$\chi(c,\overline{v})=\frac{1}{1-cd(\overline{v})}=\frac{1}{1-c(l+\tau\overline{v})}. \tag{4.2}$$

Like the scattering amplitude $(1-P)$, the factor χ is due to a mean-field effect and thus depends on macroscopic quantities. In the following a modified cross section is defined by

$$\sigma=\chi(c,\overline{v})[1-P(c)]. \tag{4.3}$$

If $c\rightarrow\hat{c}$ then $\chi\rightarrow\infty$, i.e., the cross section σ is sharply increased.

B. Corrected Boltzmann integral

1. Distribution functions

Here we consider the effect that due to the extension of the vehicles the distribution functions in the Boltzmann ansatz have to be evaluated at different positions. The vehicle causing a faster one to slow down is the distance $d(v)=l+\tau v$ ahead, where v is the velocity of the faster car before the collision. The corrected Boltzmann scattering term of Eq. (2.16) now reads

$$\int_{v}^{+\infty}dv'\sigma(v'-v)f(x+d(v'),v,t)g(x,v',w,t) - \int_{0}^{v}dv'\sigma(v-v')g(x,v,w,t)f(x+d(v),v',t). \tag{4.4}$$

Here, the cross section σ is taken at the position x. Expanding the distribution function in a Taylor series in d around x and keeping only the first two terms, one finds, for the complete Boltzmann equation,

$$\begin{aligned}\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}\right)g+\frac{\partial}{\partial v}\left(\frac{w-v}{T}g\right) &= \sigma\Bigg[\int_{v}^{+\infty}dv'(v'-v)fg'-\int_{0}^{v}dv'(v-v')gf' \\ &\quad +l\left(\int_{v}^{+\infty}dv'(v'-v)(\partial_x f)g' - \int_{0}^{v}dv'(v-v')g(\partial_x f')\right) \\ &\quad +\tau\left(\int_{v}^{+\infty}dv'(v'-v)v'(\partial_x f)g' - \int_{0}^{v}dv'(v-v')vg(\partial_x f')\right)\Bigg],\end{aligned} \tag{4.5}$$

with $f'\equiv f(x,v',t)$ and $g'\equiv g(x,v',w,t)$.

The derivation of the cumulant equations is rather lengthy and is therefore only sketched in Appendix B. The results are the following.

(i) It is easy to check that the continuity equation still holds and that for the pure moments in w no additional terms are produced.

(ii) As an additional term for the RHS of the mean velocity evolution equation (3.15) one gets

$$\alpha_1\partial_x c+\alpha_2\partial_x\overline{v}+\alpha_3\partial_x\Theta_{vv}. \tag{4.6}$$

(iii) Integration of (4.5) over $\int dv\ dw\ v^2$ leads to the following correction to the RHS of the variance equation (3.22):

$$\beta_1\partial_x c+\beta_2\partial_x\overline{v}+\beta_3\partial_x\Theta_{vv}. \tag{4.7}$$

(iv) Eventually, one finds as a correction for the RHS of the mixed cumulant equation (3.24)

$$\gamma_1\partial_x c+\gamma_2\partial_x\overline{v}+\gamma_3\partial_x\Theta_{vv}. \tag{4.8}$$

For the detailed form of the coefficients see Appendix B. They are functions of σ, l, τ and the dynamical variables $c,\overline{v},\overline{w},\Theta_{vv},\Theta_{vw}$. These new terms modify the gradient terms and change the characteristic speeds. Note that a Taylor expansion in Eq. (4.5) up to second order would also yield additional second-order derivatives of the dynamical quantities.

2. Cross section

Up to now, we have taken the scattering probability at the position x and not at the position of the actual collision. Since our model, and the underlying Boltzmann-like equation by Paveri-Fontana, is a quasi-one-dimensional model there is no left or right lane. When a fast car reaches a slow car there is a probability of passing without interaction (or of driving ''through'' the slow car). This probability depends on macroscopic quantities. In order to take this into account, one assumes that the probability of passing is just a functional of the density c, $P=P[c(x,t)]$. The modified scattering probability σ is then determined by the density and the mean velocity at the position of the slower car, i.e., at $x+d(v)$, where v is the velocity of the faster car. Expanding σ in a Taylor series, one gets the following additional first-order term for the Boltzmann equation (4.5):

$$\partial_x\sigma\left[\int_v^{+\infty}dv'(l+\tau v')(v'-v)fg' -\int_0^v dv'(l+\tau v)(v-v')gf'\right]. \tag{4.9}$$

For the derivative of σ with respect to x we find

$$\begin{aligned}\partial_x\sigma&=(1-P)\chi^2[d(\overline{v})\partial_x c+c\,\tau\partial_x\overline{v}]-\chi P'\partial_x c\\ &=\eta\partial_x c+\zeta\partial_x\overline{v},\end{aligned} \tag{4.10}$$

with $\eta=[\sigma d(\overline{v})-P']\chi$, $\zeta=\sigma\chi c\tau$, and P' denotes the derivative of P with respect to its argument. Note that $\eta,\zeta\geqslant 0$.

By calculating the moments of the integral part of Eq. (4.9), one finds as additional coefficients

$$\alpha_4=-c\Theta_{vv}(l+\tau\overline{v})-\tau\frac{2}{\sqrt{\pi}}c\Theta_{vv}\sqrt{\Theta_{vv}} \tag{4.11}$$

for the velocity equation,

$$\beta_4=-\tau c\Theta_{vv}^2 \tag{4.12}$$

for the equation of the variance Θ_{vv}, and

$$\gamma_4=-\frac{2}{\sqrt{\pi}}c\Theta_{vw}\sqrt{\Theta_{vv}}(l+\tau\overline{v})-2\tau c\Theta_{vv}\Theta_{vw} \tag{4.13}$$

for the equation of the covariance Θ_{vw}. A velocity- or variance-dependent passing probability would certainly lead to additional coefficients in the gradient terms of these quantities.

Thus the whole set of partial differential equations now reads

$$\frac{\partial}{\partial t}c+\frac{\partial}{\partial x}(c\overline{v})=0, \tag{4.14}$$

$$\partial_t\overline{v}+\overline{v}\partial_x\overline{v}=\left(\alpha_1+\eta\alpha_4-\frac{\Theta_{vv}}{c}\right)\partial_x c+(\alpha_2+\zeta\alpha_4)\partial_x\overline{v} +(\alpha_3-1)\partial_x\Theta_{vv}+\frac{\overline{w}-\overline{v}}{T}-\sigma c\Theta_{vv}, \tag{4.15}$$

$$\partial_t\overline{w}+\overline{v}\partial_x\overline{w}=-\frac{\Theta_{vw}}{c}\partial_x c-\partial_x\Theta_{vw}, \tag{4.16}$$

$$\partial_t\Theta_{vv}+\overline{v}\partial_x\Theta_{vv}=(\beta_1+\eta\beta_4)\partial_x c+(\beta_2+\zeta\beta_4-2\Theta_{vv})\partial_x\overline{v} +\beta_3\partial_x\Theta_{vv}+\frac{2}{T}(\Theta_{vw}-\Theta_{vv}), \tag{4.17}$$

$$\partial_t\Theta_{ww}+\overline{v}\partial_x\Theta_{ww}=-2\Theta_{vw}\partial_x\overline{w}, \tag{4.18}$$

$$\partial_t\Theta_{vw}+\overline{v}\partial_x\Theta_{vw}=(\gamma_1+\eta\gamma_4)\partial_x c+(\gamma_2+\zeta\gamma_4-\Theta_{vw})\partial_x\overline{v} -\Theta_{vv}\partial_x\overline{w}+\gamma_3\partial_x\Theta_{vv}+\frac{1}{T}(\Theta_{ww}-\Theta_{vw}) -\sigma\frac{2}{\sqrt{\pi}}c\Theta_{vw}\sqrt{\Theta_{vv}}. \tag{4.19}$$

Let us compare Eqs. (4.14)–(4.19) to Eqs. (3.6), (3.15), (3.16), and (3.22)–(3.24). The continuity equation holds in both systems. Since no relation between the finite car length and the desired velocity has been assumed, the equations for the mean desired velocity and for the variance Θ_{ww} remain unchanged [(3.16)=(4.16) and (3.23)=(4.18)]. The cross section $(1-P)$ has changed to the modified cross section $\sigma=\chi(1-P)$. New gradient terms appear in the other equations (4.15), (4.17), and (4.19). Let us first investigate Eq. (4.15). Owing to the sign of the coefficients, the density gradient term and the variance gradient term have an enhancing effect on the formation of a cluster, while in the coefficients of the velocity gradient both signs appear (for the detailed form of the coefficient see Appendix B). For the variance equation (4.17) similar considerations lead to an enhancing effect of the gradient terms except for the density gradient. Looking at the covariance equation (4.19), we find an overall enhancing effect, except for the velocity gradient, where again both signs appear. We remark that the equilibrium relations for the corrected system are almost the same as for the uncorrected system [Eqs. (3.25)–(3.27)]; the cross section $(1-P)$ has just been replaced by σ. Note that for the equilibrium state if $\overline{w}\neq 0$, $c\to\hat{c}$, and $\overline{v}\to 0$, then $\Theta_{vv}\to 0$ since $\sigma\to\infty$, meaning that in this limit the variance vanishes.

C. Characteristic velocities

To get an idea how the characteristic velocities change due to the new gradient terms, we consider the simpler system

$$\frac{\partial}{\partial t}c+\frac{\partial}{\partial x}(c\overline{v})=0, \tag{4.20}$$

$$\partial_t\overline{v}+\overline{v}\partial_x\overline{v}=\frac{W-\overline{v}}{T}-\frac{\Theta_{vv}}{c}\partial_x c-\partial_x\Theta_{vv}-(1-P)c\Theta_{vv}, \tag{4.21}$$

$$\partial_t\Theta_{vv}+\overline{v}\partial_x\Theta_{vv}=-2\Theta_{vv}\partial_x\overline{v}-\frac{2}{T}\Theta_{vv}. \tag{4.22}$$

These equations can be derived from the original Boltzmann equation if one assumes that all drivers have the same desired velocity W. The characteristic velocities of this system are

$$\lambda_1=\overline{v},\quad \lambda_{2,3}=\overline{v}\pm\sqrt{3\Theta_{vv}}. \tag{4.23}$$

As further simplification, we set $\tau=0$, i.e., we consider only the effect of the actual length l and we neglect the effect of the modified point of collision (Sec. IV B 2). Then the corrected version of Eqs. (4.20)–(4.22) read

$$\frac{\partial}{\partial t}c+\frac{\partial}{\partial x}(c\overline{v})=0, \tag{4.24}$$

$$\partial_t\overline{v}+\overline{v}\partial_x\overline{v}=\left(\alpha_1'-\frac{\Theta_{vv}}{c}\right)\partial_x c+\alpha_2'\partial_x\overline{v}+(\alpha_3'-1)\partial_x\Theta_{vv}+\frac{W-\overline{v}}{T}-\sigma\Theta_{vv}, \tag{4.25}$$

$$\partial_t\Theta_{vv}+\overline{v}\partial_x\Theta_{vv}=\beta_1'\partial_x c+(\beta_2'-2\Theta_{vv})\partial_x\overline{v}+\beta_3'\partial_x\Theta_{vv}-\frac{2}{T}\Theta_{vv}, \tag{4.26}$$

with

$$\alpha_1'=-\sigma l\Theta_{vv},\quad \alpha_2'=\sigma l\frac{2}{\sqrt{\pi}}c\sqrt{\Theta_{vv}},\quad \alpha_3'=-\sigma\frac{l}{2}c,$$

$$\beta_1'=0,\quad \beta_2'=-\sigma lc\Theta_{vv},\quad \beta_3'=\sigma l\frac{2}{\sqrt{\pi}}\sqrt{\Theta_{vv}}. \tag{4.27}$$

If we now calculate the characteristic velocities up to first order in l, we find

$$\Lambda_1=\overline{v}-\frac{2}{3\sqrt{\pi}}\sqrt{\Theta_{vv}}\sigma cl, \tag{4.28}$$

$$\Lambda_2=\overline{v}+\sqrt{3\Theta_{vv}}\left[1-\left(\frac{5}{3\sqrt{3\pi}}-\frac{1}{2}\right)\sigma cl\right], \tag{4.29}$$

$$\Lambda_3=\overline{v}-\sqrt{3\Theta_{vv}}\left[1+\left(\frac{5}{3\sqrt{3\pi}}+\frac{1}{2}\right)\sigma cl\right], \tag{4.30}$$

i.e., the characteristic velocities are shifted to lower values. This is intuitively quite clear since the cars have now less free space to accelerate to their desired velocities, thus the domain of influence should be smaller. We therefore assume that in the complete system (4.14)–(4.19) the characteristic velocities are lowered as well by taking into account the vehicle length.

V. NUMERICAL RESULTS

In the following we present first numerical simulations of our model, gained by stepwise numerical integration of Eqs. (4.14)–(4.19). To simplify the computation we have assumed periodic boundary conditions. The parameters are $T=30$ s, $\tau=0.5$ s, and the vehicle length is $l=5$ m. The homogeneous solution is chosen to be $c=0.8\hat{c}(\overline{v})$, $\overline{v}=26$ m/s, and $\Theta_{vv}=5.4$ m^2/s^2. The mean desired velocity and the variances are then determined from Eqs. (3.25)–(3.27) to be $\overline{w}=34$ m/s, $\Theta_{ww}=26.3$ m^2/s^2, and $\Theta_{vw}=5.4$ m^2/s^2. As initial perturbation we have added a small Gaussian peak to the otherwise constant variance Θ_{ww}, at time $t=0$ and position $x=5$ km. This corresponds to a region where some drivers desire to drive faster and some drivers desire to drive slower than in the remaining region, i.e., there is an inhomogenity in the driver behavior.

Figures 1 and 2 show the evolution of the dynamical quantities. We find that after some time a cluster builds up with a region of lower density behind the cluster [Fig. 1(a)]. The mean velocity [Fig. 1(b)] shows the corresponding behavior, i.e., the mean velocity in the cluster is lower, while after the cluster a region of higher mean velocity is found. The variation of the mean desired velocity [Fig. 1(c)] is relatively small compared to the mean velocity, but we can clearly see the separation of faster and slower vehicles due to the initial perturbation of the variance Θ_{ww}. The variance of the actual velocity [Fig. 2(a)] has a peak at the cluster region as we expect from measurements and other traffic models [11,21,24]. Looking at the evolution of the variance of the desired velocity [Fig. 2(b)] we find that the initial perturbation just propagates with the velocity $\overline{v}$ and flattens a bit. Figure 2(c) shows an interesting behavior of the covariance of the actual velocity and the desired velocity. In the region of the density cluster we find a sharp negative peak, i.e., more drivers do not drive with the velocity they actually want to, but are slowed down by the cluster. Right after the

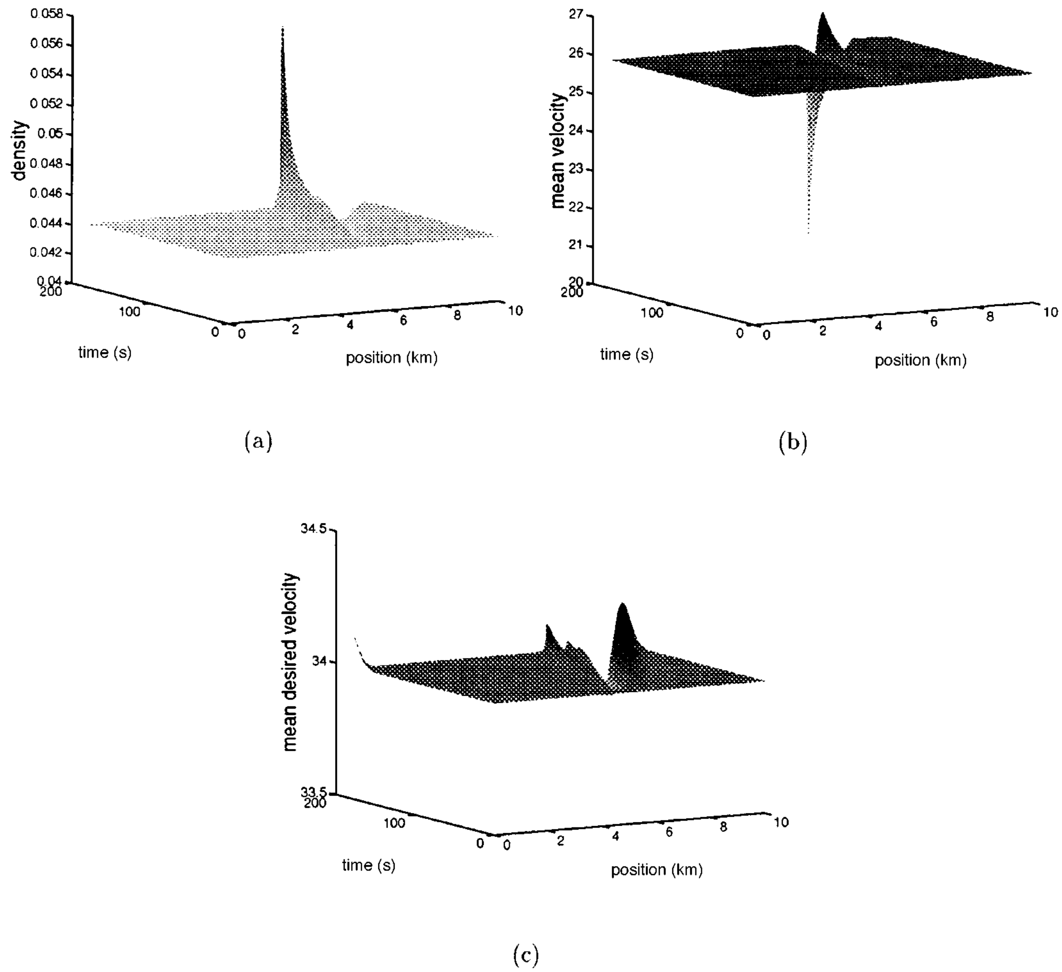

FIG. 1. Time evolution of the traffic flow starting from a homogeneous state with a small perturbation of the desired velocity variance using Eqs. (4.14)–(4.19): (a) density ρ, (b) mean velocity $\overline{v}$, and (c) mean desired velocity $\overline{w}$.

cluster there is a region with increased covariance Θ_{vw}, meaning that the drivers are free to accelerate to their desired velocity. The broad ridge on the right side of this graph corresponds to the initial perturbation in Θ_{vv} and shows that the drivers causing the onset of the cluster are not affected by it; on the contrary, there is a higher correlation between the actual velocity and desired velocity. Kerner and Kohnhäuser found a similar behavior of the density and the mean velocity, but the formation of a traffic jam has been caused by a small density perturbation. Furthermore, they observe a backward motion of the whole cluster after some time. This feature cannot yet be found in our simulation since the density has not yet reached its maximal value and the mean velocity has not been reduced to zero.

VI. CONCLUSION

In this paper, we rigorously derive a second-order traffic flow model from the microscopic level. Using a Boltzmann-like ansatz, macroscopic equations are found, similar to the hydrodynamic equations. As dynamical consequences of an additional phase-space variable on the microscopic level, the desired velocity, we find evolution equations for the mean desired velocity, for the variance of the desired velocity, and for the covariance of the actual and desired velocity. In contrast to other models, no speed-density relation is needed. The onset of traffic clusters can be explained from different driver characteristic. We have generalized Paveri-Fontana's kinetic equation to the high-density regime by taking into account the finite vehicle length. This results in gradient correction terms in the macroscopic equations, which have an overall enhancing effect on the formation of traffic clusters. A modified cross section keeps the system from reaching infinitely high densities. The relevance of the additional quantities has been demonstrated by numerical simulations where we find that a spatial variation of the variance of the desired velocity can cause the formation of a traffic jam. To evaluate this model, a stability analysis has to be made and a thorough numerical investigation has to follow. By neglecting third- and higher-order cumulants we have just derived a Euler-like traffic model. As already mentioned, one should now proceed to derive Navier-Stokes-like equations by including third-order cumulants. This would lead to viscosity terms.

APPENDIX A: INTEGRALS $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3$

In order to calculate the integrals $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3$, one has to look at a couple of other integrals first. In the following, K_n denotes the nth cumulant in v,

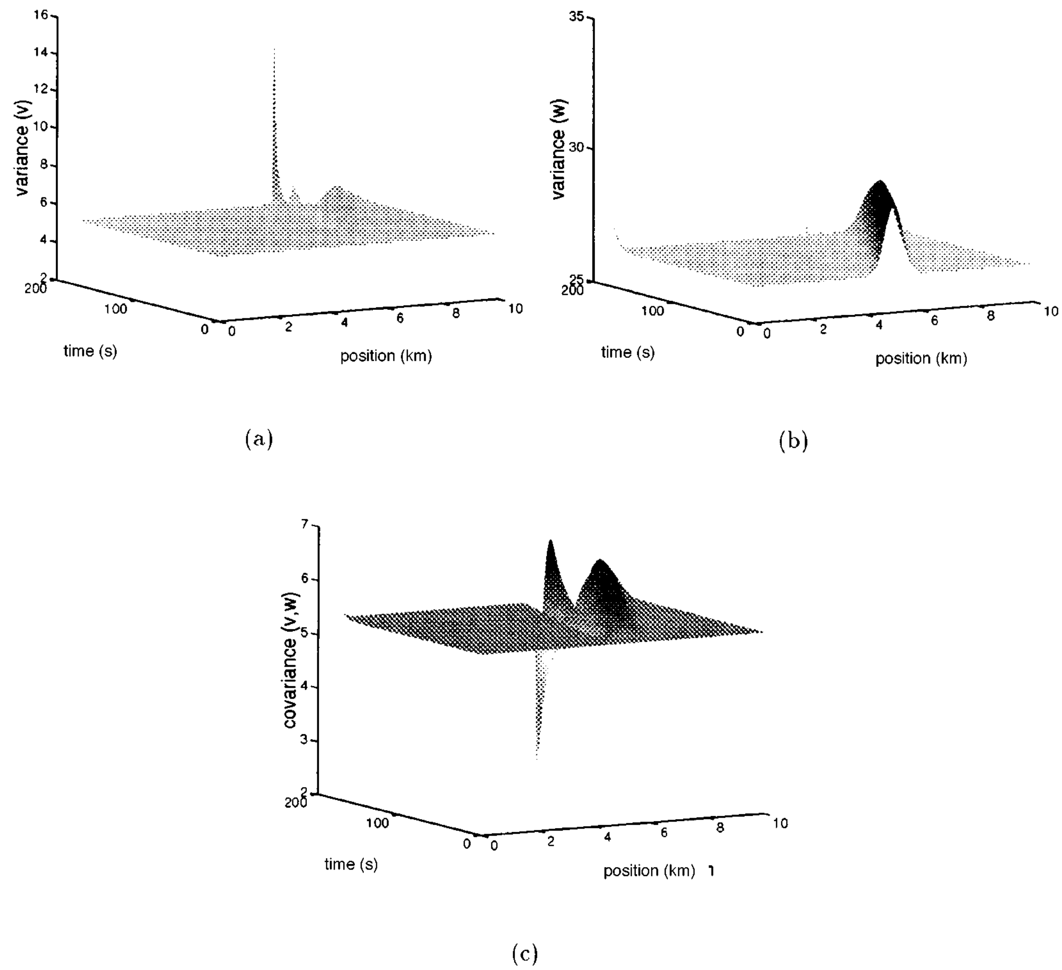

FIG. 2. Time evolution of the variances Θ_{vv}, Θ_{ww} and of the covariance Θ_{vw}: (a) variance $\overline{(\delta v)^2}$, (b) variance $\overline{(\delta w)^2}$, and (c) covariance $\overline{\delta v\,\delta w}$.

$$\int_0^{+\infty} dv\ f e^{ivz} = c\ \exp\left(\sum_{n=1}^{\infty} \frac{(iz)^n}{n!} K_n\right) \simeq c\ \exp\left(iK_1 z + \frac{i^2}{2} K_2 z^2\right), \qquad \text{(A1)}$$

where the assumption $K_n=0$ for $n \geqslant 3$ has been used. Similarly,

$$\int_0^{+\infty} dv\ v\ f e^{ivz} \simeq c(K_1 + iK_2 z)\exp\left(iK_1 z + \frac{i^2}{2} K_2 z^2\right), \qquad \text{(A2)}$$

$$\int_0^{+\infty} dv\ v^2 f e^{ivz} \simeq c(K_1K_1 + K_2 + 2iK_1K_2 z - K_2K_2 z^2)\exp\left(iK_1 z + \frac{i^2}{2} K_2 z^2\right), \qquad \text{(A3)}$$

and

$$\int_0^{+\infty} dv\ v^3 f e^{ivz} \simeq c\{3K_1K_2 + (K_1)^3 + 3i[(K_1)^2K_2 + (K_2)^2]z - 3K_1(K_2)^2 z^2 - i(K_2)^3 z^3\}\exp\left(iK_1 z + \frac{i^2}{2} K_2 z^2\right). \qquad \text{(A4)}$$

Using these relations, one easily finds

$$\int_0^{v} dv' f' = \int_0^{\infty} dv' \Theta(v - v') f' = \int_{-\infty}^{+\infty} \frac{dz}{2\pi i} \int_0^{\infty} dv' \frac{e^{iz(v-v')}}{z - i\epsilon} f' = c\int_{-\infty}^{+\infty} \frac{dz}{2\pi i} \frac{\exp[i(v-K_1)z - \frac{1}{2}K_2 z^2]}{z - i\epsilon}, \qquad \text{(A5)}$$

$$\int_0^v dv'v'f'=c\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,\frac{(K_1-iK_2z)\exp[i(v-K_1)z-\frac{1}{2}K_2z^2]}{z-i\epsilon}, \tag{A6}$$

$$\int_0^v dv'v'^2f'=c\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,\frac{(K_1)^2+K_2-2iK_1K_2z-(K_2)^2z^2}{z-i\epsilon}\,\exp[i(v-K_1)z-\tfrac{1}{2}K_2z^2], \tag{A7}$$

and

$$\int_0^v dv'v'^3f'=c\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,\frac{\{3K_1K_2+(K_1)^3-3i[(K_1)^2K_2+(K_2)^2]z-3K_1(K_2)^2z^2-i(K_2)^3z^3\}}{z-i\epsilon}\,\exp[i(v-K_1)z-\tfrac{1}{2}K_2z^2]. \tag{A8}$$

For the integral $\mathcal{I}_1$, this yields

$$\mathcal{I}_1=\int_0^{+\infty}dw\int_0^{+\infty}dv\;vwg\int_0^v dv'v'f'=c\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,\frac{(K_1-iK_2z)\exp(-iK_1z-\frac{1}{2}K_2z^2)}{z-i\epsilon}\int_0^{+\infty}dw\int_0^{+\infty}dv\;vwge^{ivz}$$
$$=c\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,\frac{(K_1-iK_2z)\exp(-iK_1z-\frac{1}{2}K_2z^2)}{z-i\epsilon}\,\frac{\partial}{i\partial z}\,\frac{\partial}{i\partial y}\bigg|_{y=0}\int_0^{+\infty}dw\int_0^{+\infty}dv\;ge^{ivz+iwy}. \tag{A9}$$

One can now again expand the integral over $dv\;dw$ in an exponential function in the cumulants up to second order

$$\int_0^{+\infty}dw\int_0^{+\infty}dv\;ge^{ivz+iwy}\simeq c\;\exp\left(iK_vz+iK_wy+\frac{i^2}{2}\,K_{vv}z^2+\frac{i^2}{2}\,K_{ww}y^2+i^2K_{vw}yz\right)$$

with an obvious notation for the cumulants in v and w. Thus

$$\mathcal{I}_1=c^2\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,[K_vK_{vw}+(K_v)^2K_w+i(K_v)^2K_{vw}z_iK_{vv}K_{vw}z+K_w(K_{vv})^2z^2+i(K_{vv})^2K_{vw}z^3]\,\frac{\exp(-K_{vv}z^2)}{z-i\epsilon}. \tag{A10}$$

For $\mathcal{I}_2$,$\mathcal{I}_3$ one gets

$$\mathcal{I}_2=\int_0^{+\infty}dw\int_0^{+\infty}dv\;wg\int_0^v dv'v'^2f'=c^2\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,[(K_v)^2K_w+K_wK_{vv}-2iK_vK_wK_{vv}z-K_w(K_{vv})^2z^2+i(K_v)^2K_{vw}z$$
$$+iK_{vv}K_{vw}K_vK_{vv}z^2-i(K_{vv})^2K_{vw}z^3]\,\frac{\exp(-K_{vv}z^2)}{z-i\epsilon}, \tag{A11}$$

and

$$\mathcal{I}_3=\int_0^{+\infty}dw\int_0^{+\infty}dv\;w\;v^2g\int_0^v dv'f'=c^2\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,[(K_v)^2K_w+K_wK_{vv}+2K_vK_{vw}+3iK_{vv}K_{vw}z+2iK_vK_wK_{vv}z$$
$$+i(K_v)^2K_{vw}z-K_w(K_{vv})^2z^2-2K_vK_{vv}K_{vw}z^2-i(K_{vv})^2K_{vw}z^3]\,\frac{\exp(-K_{vv}z^2)}{z-i\epsilon}. \tag{A12}$$

Hence, taking together the three integrals, the result now reads

$$2\mathcal{I}_1-\mathcal{I}_2-\mathcal{I}_3=c^2\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,[-2K_{vv}K_w-6iK_{vv}K_{vw}z+4(K_{vv})^2K_wz^2+4i(K_{vv})^2K_{vw}z^3]\,\frac{\exp(-K_{vv}z^2)}{z-i\epsilon}$$
$$=c^2\left(-K_{vv}K_w-\frac{2}{\sqrt{\pi}}\,\sqrt{K_{vv}}K_{vw}\right)=-c^2\left(\overline{w}\Theta_{vv}+\frac{2}{\sqrt{\pi}}\,\Theta_{vw}\sqrt{\Theta_{vv}}\right). \tag{A13}$$

APPENDIX B: CORRECTION INTEGRALS

Taking the integral $\int dv\ dw\ v$ of the Boltzmann term (4.5), the first term leads to the original term given in Eq. (3.5), while the parts proportional to l and τ can be written as

$$\sigma l\int_0^{+\infty}\int_0^{+\infty}dv\ dw\ v\left(\int_v^{+\infty}dv'(v'-v)(\partial_x f\ )g' - \int_0^v dv'(v-v')g(\partial_x f')\right)=\sigma l\int_0^{+\infty}dv\int_0^v dv'(2vv' -v^2-v'^2)(\partial_x f')f \tag{B1}$$

and

$$\sigma\tau\int_0^{+\infty}\int_0^{+\infty}dv\ dw\ v\left(\int_v^{+\infty}dv'(v'-v)v'(\partial_x f\ )g' - \int_0^v dv'(v-v')vg(\partial_x f')\right)=\sigma\tau\int_0^{+\infty}dv\int_0^v dv'(2v^2v' -v^3-vv'^2)(\partial_x f')f, \tag{B2}$$

respectively. Equation (B1) is abbreviated to

$$\sigma l(2\mathcal{H}_1-\mathcal{H}_2-\mathcal{H}_3), \tag{B3}$$

with

$$\mathcal{H}_1=\int_0^{+\infty}dv\ vf\int_0^v dv'v'(\partial_x f'), \tag{B4}$$

$$\mathcal{H}_2=\int_0^{+\infty}dv\ v^2f\int_0^v dv'(\partial_x f'), \tag{B5}$$

$$\mathcal{H}_3=\int_0^{+\infty}dv\ f\int_0^v dv'v'^2(\partial_x f'). \tag{B6}$$

To calculate $\mathcal{H}_1$ one proceeds in a similar way as in Appendix A, i.e., in the cumulant expansion of the distribution function the terms of third and higher order are neglected. Therefore

$$\begin{aligned}\mathcal{H}_1&=\int_0^{+\infty}dv\ vf\partial_x\int_0^v dv'v'f'\\ &=\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,\frac{\partial_x[c(K_1-iK_2z)\exp(-iK_1z-\frac{1}{2}K_2z^2)]}{z-i\epsilon}\\ &\quad\times\int_0^{+\infty}dv\ v\ f\ e^{ivz}\\ &=c\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,\frac{\partial_x[c(K_1-iK_2z)\exp(-iK_1z-\frac{1}{2}K_2z^2)]}{z-i\epsilon}\\ &\quad\times(K_1+iK_2z)\exp(iK_1z-\tfrac{1}{2}K_2z^2)\end{aligned} \tag{B7}$$

and similar expressions for $\mathcal{H}_2$ and $\mathcal{H}_3$.

Summing up the three integrals one finds

$$\begin{aligned}2\mathcal{H}_1-\mathcal{H}_2-\mathcal{H}_3&=\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\,[-cK_2\partial_x c-c\partial_x(cK_2)\\ &\quad+6ic^2K_2\partial_xK_1z-4i(cK_2)^2\partial_xK_1z^3]\\ &\quad\times\exp(-\tfrac{1}{2}K_2z^2)\\ &=-c\Theta_{vv}\partial_x c+\frac{2}{\sqrt{\pi}}\,c^2\sqrt{\Theta_{vv}}\partial_x\overline{v}-\tfrac{1}{2}c^2\partial_x\Theta_{vv}\,.\end{aligned} \tag{B8}$$

Analogously, for the integral proportional to τ [Eq. (B2)] the calculation yields

$$\sigma\tau\Bigg[\left(-\frac{2}{\sqrt{\pi}}\,\Theta_{vv}\sqrt{\Theta_{vv}}-\overline{v}\Theta_{vv}\right)c\partial_x c+\left(\frac{2}{\sqrt{\pi}}\,c^2\overline{v}\sqrt{\Theta_{vv}}+c^2-w\right)\partial_x\overline{v} -\frac{1}{2}\left(\frac{1}{\sqrt{\pi}}\,\sqrt{\Theta_{vv}}+\overline{v}\right)c^2\partial_x\Theta_{vv}\Bigg]. \tag{B9}$$

Taking together these two terms and dividing by c, the additional terms for the RHS of Eq. (3.15) are

$$\alpha_1\partial_x c+\alpha_2\partial_x\overline{v}+\alpha_3\partial_x\Theta_{vv}\,, \tag{B10}$$

with

$$\alpha_1=-\sigma\left[l+\tau\left(\frac{2}{\sqrt{\pi}}\,\sqrt{\Theta_{vv}}+\overline{v}\right)\right]\Theta_{vv}\,, \tag{B11}$$

$$\alpha_2=\sigma\left(\frac{2}{\sqrt{\pi}}\,\sqrt{\Theta_{vv}}(l+\tau\overline{v})+\tau\Theta_{vv}\right)c, \tag{B12}$$

$$\alpha_3=-\sigma\,\frac{1}{2}\left[l+\tau\left(\frac{1}{\sqrt{\pi}}\,\sqrt{\Theta_{vv}}+\overline{v}\right)\right]c. \tag{B13}$$

Integration of (4.5) over $\int dv\ dw\ v^2$ leads, besides the uncorrected term, first to

$$\sigma\left[l\int_0^{+\infty}\int_0^{+\infty}dv\ dw\ v^2\left(\int_v^{+\infty}dv'(v'-v)(\partial_x f\ )g'-\int_0^v dv'(v-v')g(\partial_x f')\right)+\tau\int_0^{+\infty}\int_0^{+\infty}dv\ dw\ v^2\right.$$
$$\left.\times\left(\int_v^{+\infty}dv'(v'-v)v'(\partial_x f\ )g'-\int_0^v dv'(v-v')vg(\partial_x f')\right)\right]$$
$$=\sigma\times\left[l\int_0^{+\infty}dv\int_0^v dv'(vv'^2-v'^3-v^3+v^2v')(\partial_x f')f+\tau\int_0^{+\infty}dv\int_0^v dv'(v^2v'^2-vv'^3-v^4+v^3v')(\partial_x f')f\right] \tag{B14}$$

and then to

$$\sigma\left\{\left[l(-2\overline{v})+\tau\left(-\Theta_{vv}-2\overline{v^2}-\frac{4}{\sqrt{\pi}}\,\overline{v}\sqrt{\Theta_{vv}}\right)\right]c\Theta_{vv}\partial_x c+\left[l\left(\frac{4}{\sqrt{\pi}}\,\overline{v}\sqrt{\Theta_{vv}}-\Theta_{vv}\right)+\tau\left(\overline{v}\Theta_{vv}+\frac{4}{\sqrt{\pi}}\,\overline{v^2}\sqrt{\Theta_{vv}}\right)\right]\right.$$
$$\left.\times c^2\partial_x\overline{v}+\left[l\left(\frac{2}{\sqrt{\pi}}\,\sqrt{\Theta_{vv}}-\overline{v}\right)+\tau\left(\frac{1}{2}\,\Theta_{vv}-\overline{v^2}+\frac{1}{\sqrt{\pi}}\,\overline{v}\sqrt{\Theta_{vv}}\right)\right]c^2\partial_x\Theta_{vv}\right\}. \tag{B15}$$

When proceeding to derive the equation for the variance Θ_{vv}, we have to insert at a certain point the corrected mean velocity equation. Therefore, we eventually find as corrections to the RHS of the variance equation (3.22)

$$\beta_1\partial_x c+\beta_2\partial_x\overline{v}+\beta_3\partial_x\Theta_{vv}, \tag{B16}$$

with

$$\beta_1=-\sigma\tau\Theta_{vv}^2, \tag{B17}$$

$$\beta_2=-\sigma c\Theta_{vv}(l+\tau\overline{v}), \tag{B18}$$

$$\beta_3=\sigma\left(\frac{2}{\sqrt{\pi}}\,\sqrt{\Theta_{vv}}(l+\tau\overline{v})+\tau\,\frac{1}{2}\,\Theta_{vv}\right)c. \tag{B19}$$

Analogously, the integration over $\int dv\ dw\ vw$ for the mixed moment yields

$$\sigma\left[l\int_0^{+\infty}\int_0^{+\infty}dv\ dw\ vw\left(\int_v^{+\infty}dv'(v'-v)(\partial_x f\ )g'-\int_0^v dv'(v-v')g(\partial_x f')\right)+\tau\int_0^{+\infty}\int_0^{+\infty}dv\ dw\ vw\right.$$
$$\left.\times\left(\int_v^{+\infty}dv'(v'-v)v'(\partial_x f\ )g'-\int_0^v dv'(v-v')vg(\partial_x f')\right)\right]$$
$$=\sigma\left[l\int_0^{+\infty}dv\int_0^{+\infty}dw\ wg\int_0^v dv'(2vv'-v^2-v'^2)(\partial_x f')+\tau\int_0^{+\infty}dv\int_0^{+\infty}dw\ wg\int_0^v dv'(2v^2v'-vv'^2-v^3)(\partial_x f')\right], \tag{B20}$$

and after a rather lengthy calculation one finds as the correction for the mixed moment equation (3.24)

$$\left\{\left[l\left(-\overline{w}\Theta_{vv}-\frac{2}{\sqrt{\pi}}\,\Theta_{vw}\sqrt{\Theta_{vv}}\right)+\tau\left(-\overline{vw}\Theta_{vv}-\Theta_{vw}\Theta_{vv}-\frac{2}{\sqrt{\pi}}\,\overline{w}\Theta_{vv}\sqrt{\Theta_{vv}}-\frac{2}{\sqrt{\pi}}\,\overline{v}\Theta_{vw}\sqrt{\Theta_{vv}}\right)\right]c\,\partial_x c\right.$$
$$+\left[l\left(\Theta_{vw}+\frac{2}{\sqrt{\pi}}\,\overline{w}\sqrt{\Theta_{vv}}\right)+\tau\left(\overline{v}\Theta_{vw}+\overline{w}\Theta_{vv}+\frac{3}{\sqrt{\pi}}\,\Theta_{vw}\sqrt{\Theta_{vv}}+\frac{2}{\sqrt{\pi}}\,\overline{vw}\sqrt{\Theta_{vv}}\right)\right]c^2\partial_x\overline{v}+\left[l\left(-\frac{1}{2}\,\overline{w}-\frac{1}{2\sqrt{\pi}}\,\frac{\Theta_{vw}}{\sqrt{\Theta_{vv}}}\right)\right.$$
$$\left.\left.+\tau\left(-\frac{1}{2}\,\Theta_{vw}-\frac{1}{2}\,\overline{vw}-\frac{1}{2\sqrt{\pi}}\,\overline{w}\sqrt{\Theta_{vv}}-\frac{1}{2\sqrt{\pi}}\,\overline{v}\,\frac{\Theta_{vw}}{\sqrt{\Theta_{vv}}}\right)\right]c^2\partial_x\Theta_{vv}\right\}. \tag{B21}$$

Using again the corrected mean velocity equation, we obtain as the correction for the RHS of covariance equation (3.24)

$$\gamma_1\partial_x c+\gamma_2\partial_x\overline{v}+\gamma_3\partial_x\Theta_{vv}, \tag{B22}$$

with

$$\gamma_1 = -\sigma\left(\frac{2}{\sqrt{\pi}}\,\Theta_{vw}\sqrt{\Theta_{vv}}(l+\tau\overline{v})+\tau\Theta_{vw}\Theta_{vv}\right), \quad \text{(B23)}$$

$$\gamma_2 = \sigma\left((l+\tau\overline{v})\Theta_{vw}+\tau\,\frac{3}{\sqrt{\pi}}\,\Theta_{vw}\sqrt{\Theta_{vv}}\right)c, \quad \text{(B24)}$$

$$\gamma_3 = -\sigma\left(\frac{1}{2\sqrt{\pi}}\,\frac{\Theta_{vw}}{\sqrt{\Theta_{vv}}}\,(l+\tau\overline{v})+\frac{1}{2}\,\tau\Theta_{vw}\right)c. \quad \text{(B25)}$$

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[1] S. L. Paveri-Fontana, Trans. Res. **9**, 225 (1975).

[2] M. J. Lighthill and G. B. Whitham, Proc. R. Soc. London Ser. A **229**, 317 (1955); G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).

[3] P. Michalopoulos, P. Yi, and A. Lyrintzis, Trans. Res. B **27**, 315 (1993).

[4] K. Nagel and M. Schreckenberg, J. Phys. (France) I **2**, 2221 (1992).

[5] M. Schreckenberg and A. Schadschneider, J. Phys. A **26**, L679 (1993).

[6] O. Biham, A. A. Middleton, and D. Levine, Phys. Rev. A **46**, 6124 (1992).

[7] T. Nagatani, Phys. Rev. E **48**, 3290 (1993).

[8] I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic* (Elsevier, New York, 1971).

[9] W. F. Phillips, Utah State University Report No. DOT/RSPD/DPB/50-77/17, 1977 (unpublished); Utah State University Report No. DOT-RC-82018, 1987 (unpublished).

[10] E. Alberti and G. Belli, Trans. Res. **12**, 33 (1978).

[11] D. Helbing, Physica A **219**, 375 (1995); **219**, 391 (1995); Phys. Rev. E **51**, 3164 (1995).

[12] D. Helbing, Phys. Rev. E **53**, 2366 (1996).

[13] P. Nelson, Transp. Theory Stat. Phys. **24**, 383 (1995).

[14] L. D. Landau and E. M. Lifshitz, *Lehrbuch der Theoretischen Physik* (Akademie-Verlag, Berlin, 1984), Bands V and X.

[15] J. Keizer, *Statistical Thermodynamics of Nonequilibrium Processes* (Springer, Berlin, 1987).

[16] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1952).

[17] P. P. J. M. Schram, *Kinetic Theory of Gases and Plasmas* (Kluwer Academic, Dordrecht, 1991).

[18] P. K. Munjal and L. A. Pipes, Transp. Res. **5**, 241 (1971).

[19] F. Pampel, *Ein Beitrag zur Berechnung der Leistungsfähigkeit von Strassen. Forschungsarbeiten aus dem Strassenwesen* (Kirschbaum, Bielefeld, 1955).

[20] A. Alvarez, J. J. Brey, and J. M. Casado, Transp. Res. B **24**, 193 (1990).

[21] R. Kühne, in *Highway Capacity and Level of Service*, edited by U. Brannolte (Balkema, Rotterdam, 1991), p. 211, in *Proceedings of the Ninth International Symposium on Transportation and Traffic Theory*, edited by I. Volmuller and R. Hamerslag (VNU Sciences, Utrecht, 1984).

[22] B. Kerner and P. Kohnhäuser, Phys. Rev. E **48**, R2335 (1993).

[23] P. Resibois and M. de Leener, *Classical Kinetic Theory of Fluids* (Wiley, New York, 1977).

[24] H. Zackor, R. Kühne, and W. Balz, *Untersuchungen des Verkehrsablaufs im Bereich der Leistungsfähigkeit und bei Instabilem Fluß, Forschung Strassenbau und Strassenverkehrstechnik* (Bundesministerium für Verkehr, Bonn, 1988), p. 524.